Coset and Its Properties

Left coset and Right coset :

Definition :- Let G be a group and H be a subgroup of G.

Let $H = \{h_1, h_2, h_3, \dots, h_i, \dots\}$

Take any $a \in G$. Then the collection $\{ah_1, ah_2, ah_3, \dots, ah_i, \dots\}$ is called left coset of H in G generated by a and is denoted by aH.

i.e. $aH = \{ah_1, ah_2, ah_3, \dots, ah_i, \dots\}$

i.e. $aH = \{ah_i : h_i \in H\}$

similarly.

The collection $Ha = \{h_1a, h_2a, h_3a, \dots, h_ia, \dots\}$ i.e. $Ha = \{ha : h \in H\}$ obtained by multiplying each of elements of H with a is called right coset of H in G generated by a

If G is additive group. Then the left coset of H in G generated by a is defined as

$$a+H = \{a+h : h \in H\}$$

similarly the right coset generated by 'a' is

$$H+a = \{h+a : h \in H\}.$$

Note ① If G is abelian, then the left and right cosets of H in G coincide.

Theorem:- If H is a Subgroup of the group G, then
(i) $eH = He = H$, where e is the identity
(ii) $hH = Hh = H$, where $h \in H$.

Proof:-
(i) Since e is the identity of G
we have, $eH = H$ and so also $He = H$

$\therefore eH = He = H$
i.e. H is both a left Coset and a right Coset of H in G.

(ii) Since, H is a ~~Subgroup~~ (Subgroup)
Therefore, H is closed w.r.to multiplication
Hence, the product of elements of H by h will be the elements of H in some order.

Thus if any element $a \in G$ also belongs to H.
Then, $aH = H$
$\therefore$ ~~The the Hence the proof~~ Proved.

Similarly The theorem is also true for right Coset.

Example: Let G be the additive group $I/\{4\}$ of integers modulo 4, i.e. $I/4 = [\{0\}, \{1\}, \{2\}, \{3\}]$

Soln:- Let H be the Subgroup $[\{0\}, \{2\}]$
The left Cosets of H
$\{0\} \oplus H = [\{0+0\}, \{0+2\}] = [\{0\}, \{2\}] = H$
$\{1\} \oplus H = [\{1+0\}, \{1+2\}] = [\{1\}, \{3\}]$
$\{2\} \oplus H = [\{2+0\}, \{2+2\}] = [\{2\}, \{4\}] = [\{2\}, \{0\}] = H$
$\{3\} \oplus H = [\{3+0\}, \{3+2\}] = [\{3\}, \{5\}] = [\{3\}, \{1\}]$
Two left Cosets of $H = [\{0\}, \{2\}]$ ~~or~~ $[\{1\}, \{3\}]$ i.e. H or $[\{1\}, \{3\}]$

Langrage's theorem

Statement: The order of each subgroup of a finite group G is a divisor of the order of group G.

OR,

If G is a finite group of order n and H is a subgroup of order m then m divides n.

Proof:- Let G be a group of finite order n.
Let H be a subgroup of G and let $O(H) = m$.
We have to prove that m divides n.

Since $O(H) = m$,
Suppose that $h_1, h_2, h_3, \dots, h_m$ are the m members of H.

Let $a \in G$. Then, aH is a left coset of H in G
we have, $aH = \{ah_1, ah_2, ah_3, \dots, ah_m\}$

All the elements of aH are distinct.
Suppose two of them are equal
Let $ah_i = ah_j$
$\Rightarrow h_i = h_j$ {Left Cancellation law}
which contradicts the fact that all the elements of H are distinct.
Hence, $ah_i \neq ah_j$
Therefore, each left coset of H in G has m distinct members.
Any two distinct left cosets of H in G are disjoint.

Since, G is a finite group.
$\therefore$ the number of distinct left cosets of H in G will be finite say k.

Also, the union of these K distinct left cosets of H in G.

Thus if $a_1H, a_2H, \ldots, a_KH$ are the K distinct left cosets of H in G.

Then, $G = a_1H \cup a_2H \cup \cdots \cup a_KH$ — (1)

Two distinct left cosets are mutually disjoint and each left coset has m elements.

Hence the total number of elements in the union of K disjoint left cosets will be mK

This will be equal to the number of elements in G.

Hence, $n = mK \Rightarrow \frac{n}{m} = K$

$\Rightarrow m$ is divisor of n

$\Rightarrow O(H)$ is a divisor of $O(G)$

Hence Proved